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Accuracy of SPH viscous flow models

David I. Graham*, † and Jason P. Hughes

School of Mathematics and Statistics, University of Plymouth, Plymouth PL4 8AA, U.K.

SUMMARY

In this paper, we quantify how the accuracy of 1D and 2D smoothed particle hydrodynamics simulations of viscous diffusion depend upon (i) the mean inter-particle distance Δx, (ii) the smoothing length h and (iii) the randomness of the particle positions. In both the 1D and 2D cases, the method converges only in the case where randomness is absent and for a few values of $h/\Delta x$ including both integer (for 1D) and non-integer values (both 1D and 2D). For any other (fixed) value of $h/\Delta x$, the method does not converge. In most cases, increasing randomness decreases the accuracy of the results, as does the ability of particles to move. Simulations using larger values of $h/\Delta x$ appear to be less influenced by particle randomness and ability to move. Copyright © 2007 John Wiley & Sons, Ltd.

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1. INTRODUCTION

Errors in smoothed particle hydrodynamics (SPH) simulations can behave in a strange way for those of us who are more familiar with more traditional grid-based numerical schemes. In particular, Quinlan *et al.* [1] showed that the accuracy of SPH approximations can deteriorate when resolution is improved by increasing the number of computational particles. The issue is due to the fact that there are two independent length scales in SPH—namely, the inter-particle distance Δx and the kernel-smoothing distance h. SPH simulations usually 'converge' only when both of these quantities reduce to zero and when Δx diminishes more rapidly than h [2].

1.1. SPH approximations

SPH is a mesh-free method in which flow variables are approximated by a kernel-smoothing interpolation based on values stored at computational particles, and differentiation is in principle

*Correspondence to: David I. Graham, School of Mathematics and Statistics, University of Plymouth, Plymouth PL4 8AA, U.K.
†E-mail: dgraham@plymouth.ac.uk

performed on the kernel rather than any underlying grid. The (smoothed) value of any flow variable F at any point a is given by

$$\langle F\rangle_a = \sum_b \frac{m_b}{\rho_b} F_b W_{ab} \tag{1}$$

where the sum is over all neighbouring particles within the radius of the interpolating kernel, m_b, ρ_b and F_b are the values of mass, density and F at particle b, W_{ab} is the value of the interpolation function $W(\mathbf{r}_a - \mathbf{r}_b, h)$, with separation $\mathbf{r}_a - \mathbf{r}_b$ and characteristic kernel dimension h.

The (smoothed) gradient of F could be found by direct differentiation of Equation (1). In practice, however, the following approximation is usually used:

$$\langle \nabla F\rangle_a = \frac{1}{\rho_a} \sum_b m_b F_{ba} \nabla W_{ab} \tag{2}$$

where $F_{ba} = F_b - F_a$, $\mathbf{r}_{ab} = \mathbf{r}_a - \mathbf{r}_b$ and ∇W_{ab} is the gradient of W_{ab}.

Similarly, the Laplacian could be found by directly differentiating (1). However, when low-degree polynomials are used as the interpolation functions, inaccuracies can arise, particularly spurious oscillations as observed in Cummins and Rudman [3]. Here the following 'second-order' approximation is used, similar to that used by Brookshaw [4], Morris *et al.* [5] and others:

$$\langle \nabla^2 F\rangle_a = \sum_b \frac{2m_b}{\rho_b} F_{ab} \mathbf{r}_{ab} . \frac{\nabla W_{ab}}{r_{ab}^2} \tag{3}$$

Throughout this paper, we utilize the following, widely used interpolation kernel:

$$W(\mathbf{r}, h) = C(h) \begin{cases} 1 - 3q^2/2 + 3q^3/4, & 0 \leqslant q \leqslant 1 \\ (2-q)^3/4, & 1 < q \leqslant 2 \end{cases} \tag{4}$$

where W is zero otherwise, $q = |\mathbf{r}|/h$, and $C(h) = 2/(3h)$ in 1D and $10/(7\pi h^2)$ in 2D so that W integrates to unity over a circle of radius $2h$. W approaches a delta function as h diminishes.

2. ANALYSIS OF APPROXIMATIONS

2.1. Regular spacing

In order to analyse the accuracy of (3), a Taylor series expansion of all the contributions to this expression was found. We concentrate on local short-range errors, in which case the analysis is not specific to a particular flow but does not address more specific long-range errors encountered in SPH [6]. In the first instance, particles b were uniformly distributed along the x-axis at regular intervals Δx, and particle a was located at the origin. For values of $h/\Delta x$ between 0.525 (in which case a was influenced only by particles at $\pm\Delta x$) and 5.025 (when a was influenced by 20 particles at distances up to $\pm 5\Delta x$), the contributions to (3) were summed symbolically using the Maple computer algebra system. For regularly spaced particles, the leading contribution is

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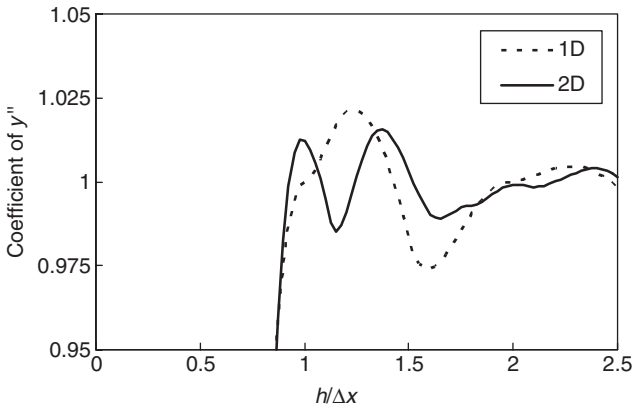

Figure 1. SPH approximation of y''.

proportional to the second derivative of F (and only even derivatives appear in the approximation) and the remaining terms are of order Δx^2. However, only for a small number of values of $h/\Delta x$ is the coefficient of the second derivative equal to unity (thus errors are $O(1)$ and the method does not converge). Figure 1 shows a plot of this coefficient for values of $h/\Delta x$ between 0.8 and 2.5 (which is consistent with and extends the results in Table I of Cleary and Monaghan [7]).

It is exactly unity at integer values of $h/\Delta x$ and also for values around 1.425, 2.45, 3.45 and 4.5 (and note that the method is actually central differencing for $h/\Delta x = 1$). For these values of $h/\Delta x$, the method would be expected to converge (so long as the spacing remains uniform) as the number of particles increases. For any other values of this ratio, the method does not converge. However, the variation of the coefficient from the value of 1 diminishes significantly as $h/\Delta x$ increases.

This exercise was then repeated to analyse the errors from 2D SPH approximations. Here, particles b were uniformly distributed in a regular lattice arrangement with spacing Δx in both directions, with particle a again located at the origin. So that ratios of $h/\Delta x$ up to 2.5 could be analysed, the contributions from up to 72 neighbouring particles were accounted for, again using Maple's symbolic capabilities. In this case, the leading contribution is proportional to the Laplacian, but again the coefficient is unity for only a few values of $h/\Delta x$, namely around 0.925, 1.25, 1.525 and 2.2. The values of the coefficient are also shown in Figure 1. For $h/\Delta x$ exceeding 1, the coefficient of the Laplacian varies slightly less than was the case in 1D, and the variation again diminishes as $h/\Delta x$ increases. Again, only even derivatives appear and higher derivatives are $O(\Delta x^2)$.

2.2. *Irregular spacing*

In practice, SPH particles are irregularly spaced, because they follow the motion of the fluid flow. The above analysis must therefore be modified to take into account the irregularity of the particle spacing. This was allowed for by perturbing the positions of neighbouring particles b from their initial regular spacing. Thus, each particle was shifted by an amount $(\delta x, \delta y)$, where δx and δy were randomly and independently sampled for each particle from a uniform distribution of mean 0

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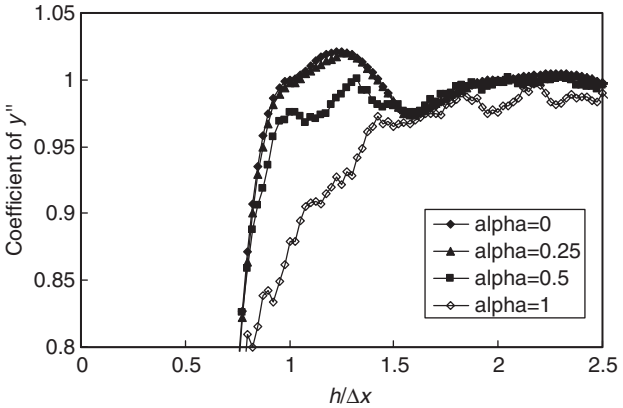

Figure 2. SPH approximation of y'' (1D, random).

and width α ($\delta y = 0$ for 1D analysis). Thus, α is a measure of randomness of the particle positions. In order to characterize the behaviour in these random configurations, the SPH expression for the Laplacian was again evaluated as a sum of Taylor series using Maple and the coefficients of the various derivatives were identified. Since the coefficients are different for every randomized particle configuration, 50 repetitions were carried out to find mean values of coefficients (this can be viewed as an attempt to mimic what happens when particles move randomly as in SPH simulations).

2.2.1. One-dimensional analysis. In the 1D case, the leading term in the SPH approximation was not the second derivative, but a first derivative. Even though the coefficient of this derivative was generally small, it was divided by Δx, so that keeping a given irregular arrangement whilst reducing the inter-particle spacing would result in a divergent approximation. In general, derivatives of all orders appear in the approximation. It appears to be randomly distributed around a mean value of zero, and the variability is largest for small values of $h/\Delta x$ and diminishes as $h/\Delta x$ increases. Figure 2 plots the coefficient of the second derivative as a function of $h/\Delta x$. For values of $h/\Delta x$ around unity, the randomized particle positions lead to significantly diminished values of this coefficient for large values of α. For larger values of $h/\Delta x$, the coefficient approaches 1 but appears to do so very slowly.

2.2.2. Two-dimensional analysis. In the 2D case, there are two leading terms, both first derivatives divided by Δx and with generally different (though small, and apparently with mean-zero) coefficients. All three second derivatives appear in the approximations. The coefficients of the xx and yy derivatives are comparable (though not identical, in general) and the coefficient of the cross-derivative term is small and has a mean value close to zero. In general, all higher-order derivatives are present in the approximations. Figure 3(a) and (b) plots the coefficient of the xx and yy derivatives as a function of $h/\Delta x$. Similar behaviour to that seen in 1D is observed: small values of the coefficient of these terms for large values of α and smaller values of $h/\Delta x$. For larger values of $h/\Delta x$, the coefficient again approaches 1 but equally as slowly as in 1D.

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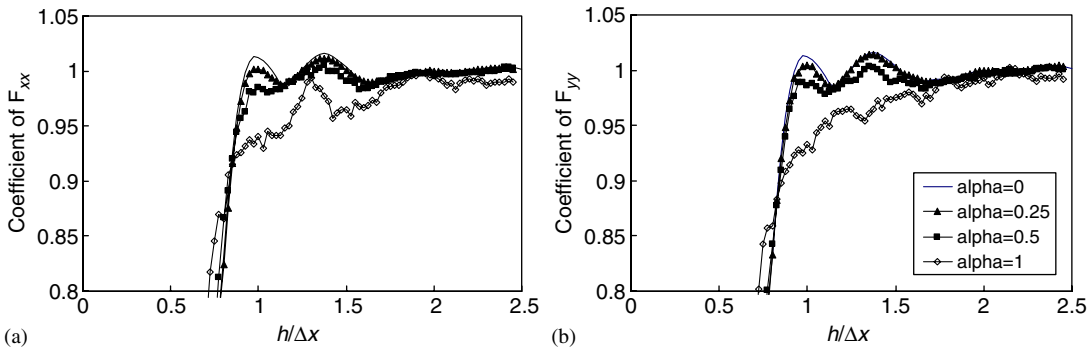

Figure 3. SPH approximation of (a) F_{xx} and (b) F_{yy}.

3. NUMERICAL RESULTS

We simulate pressure-driven flow between parallel plates as a startup problem using SPH. At $t=0$, particles are static. They are then accelerated by an instantaneously activated pressure gradient (which is implemented as a uniform body force). The resulting equation of motion is

$$\frac{\partial \mathbf{u}}{\partial t} = -\mathbf{G} + \frac{1}{Re}\nabla^2 \mathbf{u} \tag{5}$$

where $\mathbf{u}$ is the velocity, $\mathbf{G}$ is a constant vector representing the pressure gradient and Re is the Reynolds number. Without loss of generality, the distance between the plates is equal to unity, we set $Re=1$ and choose $\mathbf{G}$ to give a maximum velocity equal to 1 at the centreline in the steady state. The no-slip condition is applied on the plates, with fixed dummy particles positioned behind these wall boundaries [8] having velocity equal to the corresponding wall particle. An explicit scheme (traditional in SPH) was used to integrate [5], and the Laplacian term is evaluated using the approximations given in Section 2. Computations were carried out with both regularly and irregularly spaced particles, in both 1D and 2D, and were considered to be fully developed at 1 s (at which time the analytical solution shows a centreline u-velocity $>99.9\%$ of the steady-state value). In the computations, the timestep, $\mathrm{d}t$, satisfies the viscous diffusion timescale ($\mathrm{d}t \leqslant 0.1\Delta x^2/\nu$) and the Courant condition ($\mathrm{d}t \leqslant 0.1\Delta x/U_{\max}$). For $Re=1$ the viscous timescale is significantly smaller and so this condition dictates the timestep used. The error comparisons presented are all computed from the steady-state solution at a fixed time of 1 s; hence the number of timesteps used is different for each particle resolution. This approach is what would be used in practical applications of SPH, as the solution is sought in as few timesteps as possible in order to reduce the computational effort. However, simulations have also been carried out using a fixed number of timesteps for all resolutions. In this case the timestep was based on a resolution of 100 particles. This gave results that were virtually identical to those obtained when the timestep is set for each individual resolution.

3.1. One-dimensional results

3.1.1. Regular spacing. In 1D, we assume a flow parallel to the x-axis in an xy coordinate system, with velocity dependent on y only. Three values of $h/\Delta x$ were used, namely $h/\Delta x=1$,

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Table I. Convergence for uniformly spaced particles (1D).

n	$h/\Delta x=1$	$h/\Delta x=1.2$	$h/\Delta x=1.5$
9	4.19E−05	3.49E−02	1.72E−01
13	1.55E−05	1.37E−02	8.01E−02
21	4.27E−06	9.92E−03	4.66E−02
37	1.09E−06	1.52E−02	3.12E−02
69	2.75E−07	1.77E−02	2.38E−02
133	6.89E−08	1.89E−02	2.02E−02
261			1.84E−02

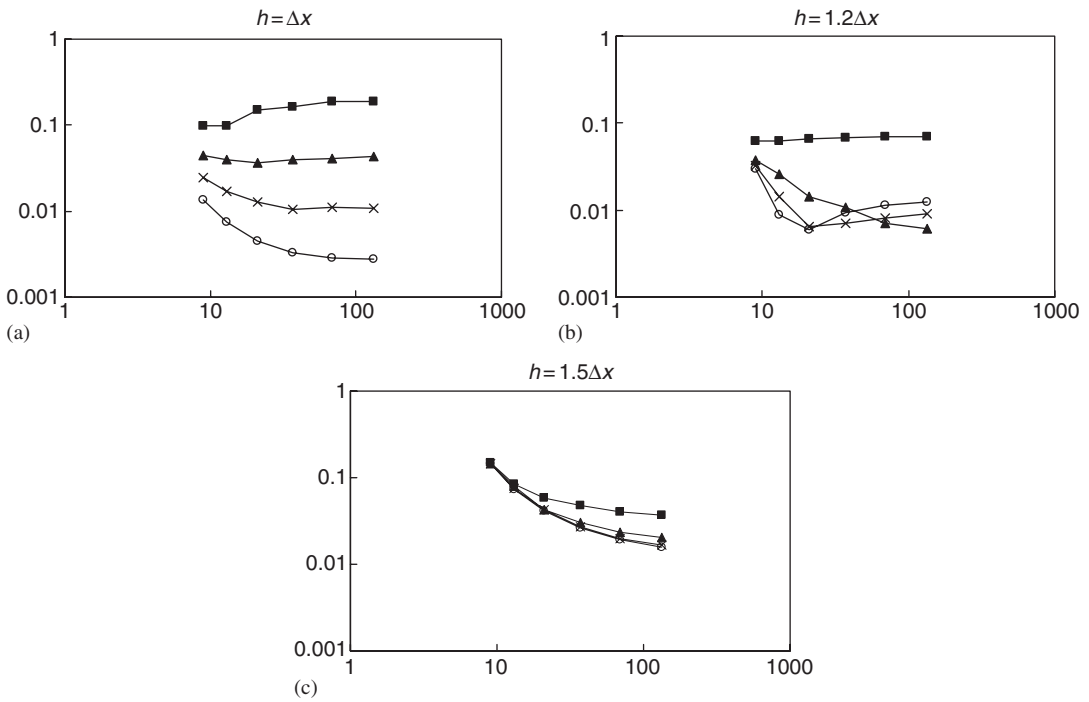

Figure 4. Errors at steady state (1D simulations) *vs* $1/\Delta x$: (a) $h=\Delta x$; (b) $h=1.2\Delta x$; and (c) $h=1.5\Delta x$. (○) $\alpha=0.0625$; (×) $\alpha=0.125$; (▲) $\alpha=0.25$; (■) $\alpha=0.5$. Randomized tests: 50 realizations.

1.2 and 1.5—these values are often found in the literature. In order to quantify error behaviour, the root mean square (r.m.s) difference between the computed values and those obtained from the analytical solution was computed. For irregular particle spacing this was repeated 50 times for each combination of h and α. As noted above, when $h=\Delta x$, the SPH formulation of the viscous term is simply central differencing thus the error in the computational results is $O(\Delta x^2)$ and errors can be reduced indefinitely by increasing the number of SPH particles in the simulation. However, for other values of $h/\Delta x$, the error behaviour is much less straightforward, as shown in Table I. In particular, for $h/\Delta x=1.2$, errors initially decrease then increase and

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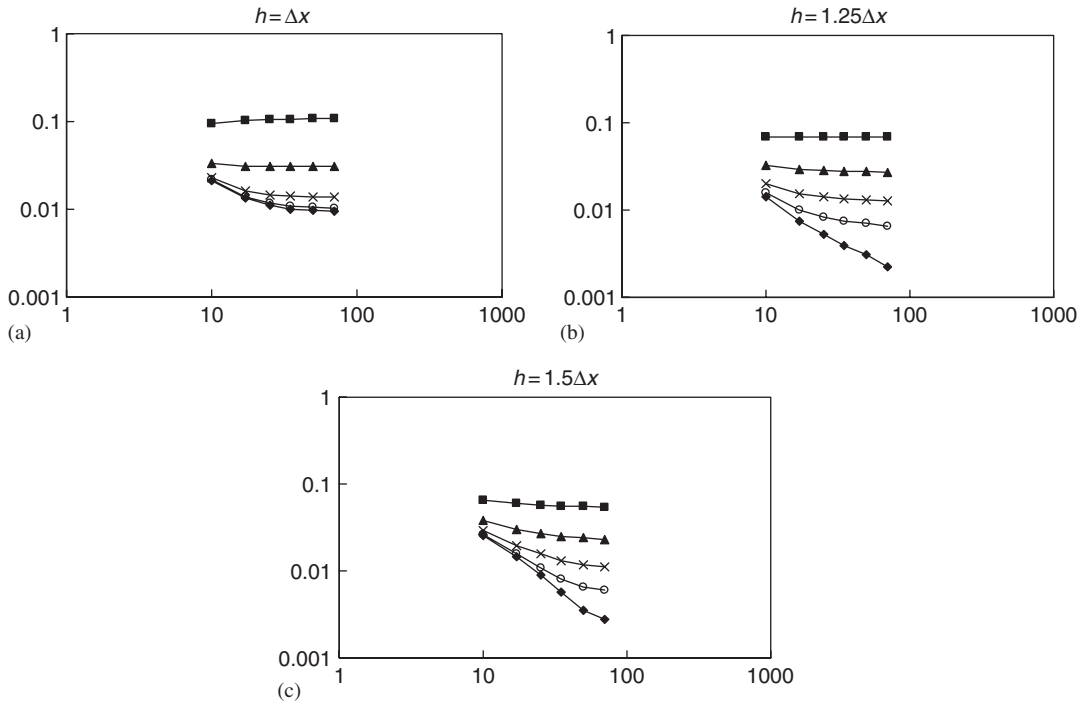

Figure 5. Errors at steady state (2D simulations, fixed particles) *vs* $1/\Delta x$: (a) $h=\Delta x$; (b) $h=1.25\Delta x$; and (c) $h=1.5\Delta x$. (♦) $\alpha=0$; (○) $\alpha=0.0625$; (×) $\alpha=0.125$; (▲) $\alpha=0.25$; (■) $\alpha=0.5$. Randomized tests: 50 realizations.

flatten off as inter-particle spacing (which is inversely proportional to the number of particles, n) decreases. For $h/\Delta x=1.5$ the errors are still decreasing but not as quickly as those for $h/\Delta x=1$. Overall, the behaviour appears to be consistent with that observed in the previous section.

3.1.2. Irregular particle spacing. In SPH computations, particle spacing is almost never uniform. To model this, SPH simulations have been undertaken in which particle positions were random. Figure 4(a)–(c) illustrates the convergence behaviour for three cases: (a) $h=\Delta x$; (b) $h=1.2\Delta x$ and (c) $h=1.5\Delta x$. In almost every case, randomness (increasing α) increases the error. However, increasing randomization can—as in case (b)—bring about a reduction in errors. This initially surprising result can be explained with reference to Figure 3 as follows. For $\alpha=0$, the coefficient of the second derivative is over-predicted (effectively increasing the viscosity in the simulation). As α increases, the coefficient becomes closer to 1, subsequently diminishing as α increases beyond 0.5. The fact that, for $h=1.5\Delta x$, errors are generally larger than for $h=\Delta x$ and $h=1.2\Delta x$ and are insensitive to small values of α is again consistent with Figure 3. In every case, errors seem to converge extremely slowly or not at all.

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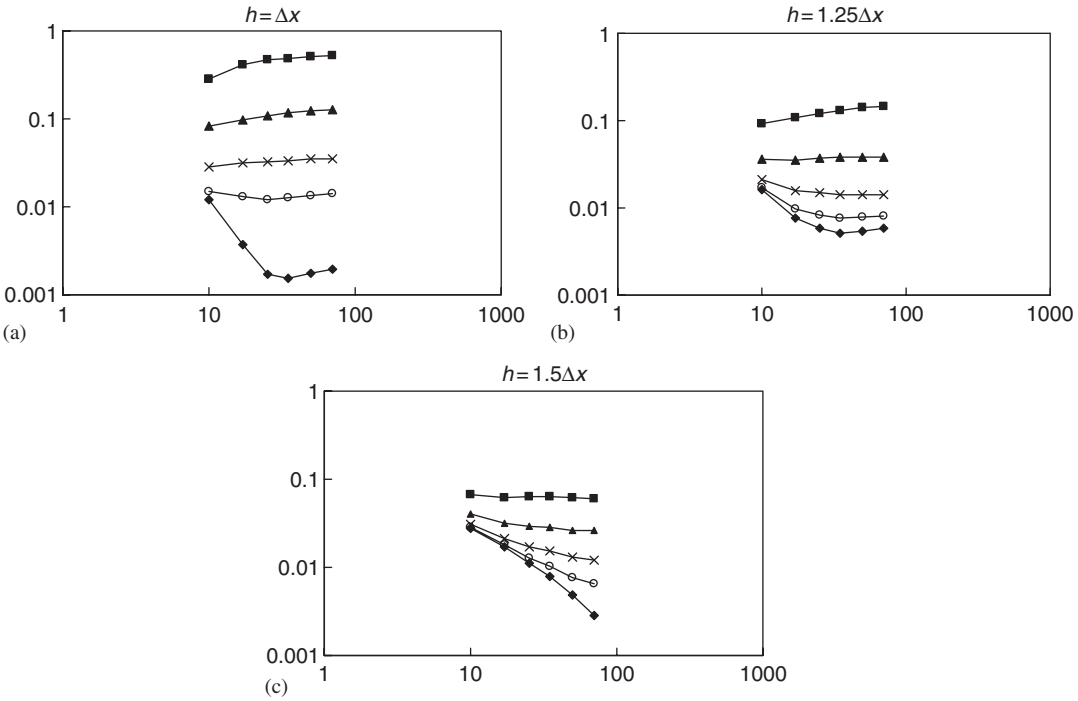

Figure 6. Errors at steady state (2D simulations, moving particles) *vs* number of particles: (a) $h=\Delta x$; (b) $h=1.25\Delta x$; and (c) $h=1.5\Delta x$. (♦) $\alpha=0$; (o) $\alpha=0.0625$; (×) $\alpha=0.125$; (▲) $\alpha=0.25$; (■) $\alpha=0.5$. Randomized tests: 50 realizations.

3.2. Two-dimensional results

In 2D, we do not assume *a priori* that the flow is unidirectional. However, we do assume here that the pressure gradient is uniform. Again, three values of $h/\Delta x$ were used, namely $h/\Delta x=1$, 1.25 and 1.5—chosen on the basis of the results from Section 2. In 2D, when $h/\Delta x=1$, the SPH formulation of the viscous term is no longer simply central differencing, due to the presence of particles located at the corner points in a 3×3 lattice. As seen in Section 2, however, for $h/\Delta x=1.25$ and 1.5, the SPH coefficient of the Laplacian is very close to 1, thus the method should be (almost) $O(\Delta x^2)$ for regularly spaced particles.

Two separate sets of 2D simulations have been completed. In the first, particles were fixed in space, contrary to usual SPH practice of moving particles according to their velocity. In the second, particles are allowed to move in the traditional SPH way. In order to quantify error behaviour, SPH approximations for u-velocity components are evaluated at 100 points along the mid-plane and the r.m.s. difference between the computed values and those obtained from the analytical solution was computed. This was repeated 50 times for each combination of h and α.

Figure 5(a)–(c) shows the mean of the 50 RMS errors for the fixed particles. Again, these results can be explained with reference to the analysis in Section 2. For regularly spaced particles with $h/\Delta x=1.25$ and 1.5, r.m.s. errors are still decreasing with n (where the mean inter-particle spacing Δx is $1/n$), whereas for $h/\Delta x=1$, errors flatten off—all consistent with Figure 3. Increasing

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randomness generally increases the errors in the approximations, again consistent with Figure 3, although perhaps the results $h/\Delta x = 1$ may have been expected to improve with moderate randomness. For the range of values of Δx used here, convergence did not reach second order even for regularly spaced particles, but this is in any case compromised by the (small) zeroth-order errors that appear.

Figure 6(a)–(c) shows the corresponding error results in the case where particles are allowed to move. For regularly spaced particles, the motion of the particles has a beneficial effect for $h/\Delta x = 1$, is detrimental for $h/\Delta x = 1.25$ and has little effect for $h/\Delta x = 1.5$. The relationship between these results and those in Section 2 is less clear, but perhaps the effect of moving the regularly spaced particles is akin to increasing randomness of fixed particles. It could then be argued that the expected benefit of moderate randomness to $h/\Delta x$ which was not seen in fixed-particle runs now becomes apparent when the particles move. When the initial particle spacing is random, the effect of moving the particles is detrimental for $h/\Delta x = 1$ and neutral for the other two cases.

4. CONCLUSIONS

Errors in both 1D and 2D SPH viscous flow models have been analysed both analytically and numerically. Truncation errors have been analysed for both regularly and irregularly spaced particles in 1D and 2D. It is shown that the method is convergent only under very unusual circumstances and that random particle configurations can have a dramatic effect on the accuracy of the SPH approximations. Errors in numerical approximations have been quantified by investigating an impulsive pressure-driven flow between parallel plates. The effects of randomness of particle position and of particle freedom-to-move have been quantified (both generally decrease accuracy). For 2D simulations, it is shown that approximations with $h/\Delta x = 1.5$ are (almost) convergent for regularly spaced particles and are less sensitive to particle randomness and motion than those with $h/\Delta x = 1$ or 1.25.

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